LECTURE 18

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Let's start with an exercise on computing flux.

Lemma 1. *Suppose that a surface S in $\mathbb{R}^3$ is given implicitly as the set $g(x,y,z) = c$ for some constant c. Then at any point $s \in S$, the gradient $\nabla g(s)$ is a normal vector at s.*

Proof. Let $\mathbf{r} : \Omega \to \mathbb{R}^3$ be a parametrization for the surface near s, where $\Omega \subseteq \mathbb{R}^2$ is some open subset. Then we have

$$\frac{\partial g}{\partial u}(s) = \nabla g(s) \cdot \mathbf{r}_u(s) = 0.$$

The same is true for $\mathbf{r}_v(s)$. Hence $\nabla g(s)$ is orthogonal to both $\mathbf{r}_u(s)$ and $\mathbf{r}_v(s)$, as desired. □

Example 2. Find the flux of $\mathbf{F} = yz\mathbf{j} + z^2\mathbf{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z \geq 0$, by the planes $x = 0$ and $x = 1$.

Solution. View S as the level surface given by the vanishing of $g(x,y,z) = y^2 - z^2 - 1$. By Lemma 1, the unit normal vector field is given (up to a sign) by

$$\mathbf{n} = \frac{\nabla g}{\|\nabla g\|} = \frac{2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{y^2 + z^2}} = y\mathbf{j} + z\mathbf{k}.$$

By drawing a picture, you can easily see that this is already pointing outward, so we do not need to adjust the sign. Now we use x, y as variables for S. Then

$$d\sigma = \frac{\|\nabla g\|}{|g_z|} dA = \frac{1}{z} dA$$

where $A = dxdy$ is the infinitesimal unit area element on the projection S' of S to the xy-plane. Putting these together, we have

$$\int_S \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_{S'} (yz\mathbf{j} + z^2\mathbf{k}) \cdot (y\mathbf{j} + z\mathbf{k}) \frac{1}{z} dA = \iint_{S'} 1 dA = \text{Area}(S') = 2.$$

■

Now let us move on to the Stokes' theorem.

Theorem 3. *Let S be a smooth oriented surface having a (piecewise) smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then*

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} d\sigma$$

where $\mathbf{T}$ is unit tangent vector field on C in the counterclockwise orientation with respect to $\mathbf{n}$.

Recall that on a surface, an orientation is equivalent to a choice of unit normal vector field $\mathbf{n}$. I find the easiest way to decide the orientation on the curve C is to use the "right hand rule". Namely, you put your thumb in the direction of $\mathbf{n}$ or some point in the interior of S, then the rest of your fingers are pointing in the "counterclockwise orientation" of the C. See the picture below.

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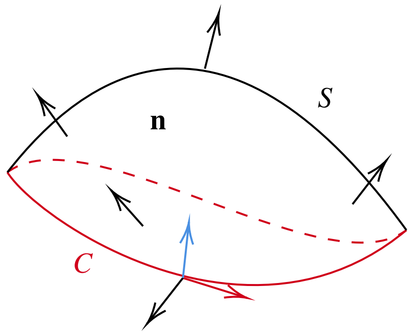

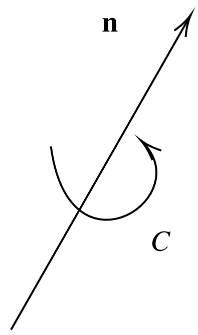

If you don't like this, you can also pick a point on the boundary curve C, then the unit tangent vector $\mathbf{T}$ given by the counterclockwise orientation (the red arrow above) and the unit normal vector $\mathbf{n}$ are related by the condition that $\mathbf{n}\times\mathbf{T}$ (the blue arrow above) is pointing inward to the surface.

Example 4. Let us verify the formula using the hemisphere S given by $x^2+y^2+z^2=9, z\geq 0$ with boundary circle $C: x^2+y^2=9, z=0$, and the vector field $\mathbf{F}=y\mathbf{i}-x\mathbf{j}$. As usual, we parameterize C using polar coordinates

$$\mathbf{r}(t)=3\cos t\,\mathbf{i}+3\sin t\,\mathbf{j},\quad 0\leq t\leq 2\pi.$$

The tangent vector is

$$\mathbf{T}=\mathbf{r}'(t)=-3\sin t\,\mathbf{i}+3\cos t\,\mathbf{j}.$$

The line integral becomes

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds=\int_0^{2\pi}-9\,dt=-18\pi.$$

Now let us compute the surface integral. First, compute the curl of $\mathbf{F}$:

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial_x & \partial_y & \partial_z\\ y & -x & 0\end{vmatrix}=-2\mathbf{k}.$$

We have

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma=\iint_{S'}(\nabla\times\mathbf{F})\cdot\frac{\nabla g}{\|\nabla g\|}\frac{\|\nabla g\|}{|g_z|}dA=\iint_{S'}\frac{(-2\mathbf{k})\cdot(2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k})}{2z}dA=\iint_{S'}-2dA=-18\pi.$$

Example 5. Suppose in the above example, we set C to be intersection of the sphere $x^2+y^2+z^2=9$ and the plane $x+y+z=0$ instead. The radius of C is still 3 (exercise). Let us compute the circulation of $\mathbf{F}$ along C with counterclockwise orientation using Stokes. In this case, we view C as the boundary of

$$S:=\{x^2+y^2+z^2\leq 9\}\cap\{x+y+z=0\}.$$

It it easy to see that

$$\mathbf{n}=\frac{1}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

on S. We note that $(\nabla\times\mathbf{F})\cdot\mathbf{n}=-2/\sqrt{3}$ is a constant, so that we conveniently compute

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,d\sigma=\frac{-2}{\sqrt{3}}\iint_S d\sigma=\frac{-2}{\sqrt{3}}\text{Area}(S)=\frac{-18\pi}{\sqrt{3}}.$$

Example 6. Let us find the circulation of the field $\mathbf{F}=(x^2-y)\mathbf{i}+4z\mathbf{j}+x^2\mathbf{k}$ around the curve C, where C is the intersection of the plane $z=2$ and the cone $z=\sqrt{x^2+y^2}$, oriented counterclockwise.

You can of course do it directly. Let C be parametrized by

$$\mathbf{r}(t)=2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}+2\mathbf{k},\quad 0\leq t\leq 2\pi.$$

Then we compute

$$\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)=-8\cos^2 t\sin t+4\sin^2 t+16\cos t.$$

Integrating this from 0 to 2π directly, we get 4π.

Now let us compute this using Stokes, viewing C as the boundary of the cone $z=\sqrt{x^2+y^2}$ below it. The orientation of the cone should be chosen so that $\mathbf{n}$ points (not straightly) upward.

Use cylindrical coordinates:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\mathbf{k}, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

Normal vector:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + r\mathbf{k}.$$

It is already pointing upwards, so we do not need to adjust the sign. Then we have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 - y & 4z & x^2 \end{vmatrix} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}.$$

Finally, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^2 (-4\mathbf{i} - 2r\cos(\theta)\mathbf{j} + \mathbf{k}) \cdot (-r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + r\mathbf{k})\, dr\, d\theta = 4\pi.$$